DEPARTMENTS.

*DISCUSSION.

THE TANGENT NORMALS TO A LIMACON.

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This problem was suggested by Dr. Glenn at the end of the solution of Problem 254, August-September, 1905.

Let the required line be $2ax+2by+c=0$, which is to be both tangent and normal to

$$(x^2+y^2+cx)^2=\frac{c^2}{e^2}(x^2+y^2).$$

Two special cases may be disposed of first: one is that of tangent lines parallel to the Y-axis, the other the case of tangents passing through the origin.

The latter is excluded from the general case by the particular value chosen for the constant term in the straight line, but it may be shown that the only line which satisfies the problem in this case is $y=0$ for $e=1$, *i. e.* the axis of the cardioid.

The former case leads to a solution when $e=\frac{3+\sqrt{5}}{2}$, the point of tangency being at the extremity of the loop, while the line is normal at both of the remaining intersections. The general solution now proceeds under the assumptions $b\neq0$, $c\neq0$, and is entirely analogous to the solution previously given, p. 157, for the case in which $e=1$.

Transform the line and limaçon by the inversion

$$x=\frac{cx'}{x'^2+y'^2},\ y=\frac{cy'}{x'^2+y'^2},$$

obtaining $x^2+y^2+2ax+2by=0$, $x^2(1-e^2)+y^2-2e^2x-e^2=0$.

Since angles are unchanged by inversion the problem is now to determine a and b so that the circle shall be both tangent and normal to the conic. The two curves have a point of orthogonal intersection when $x=-1$ or $1-2a$, but the first value leads to $x^2+y^2=0$, which is excluded. When $x=1-2a$, in which $a\neq1$ for the same reason, then y and b may be found in terms of a as a parameter.

It will simplify later work to write $a=1+h$ $(h\neq0)$, $e^2=1+d$, so that the point in question is

*The department *Discussion* will be devoted primarily to generalizations and extensions of problems solved in our columns. Some important researches have resulted from such generalizations in the past and it is believed that a department devoted to such investigations is desirable. G.

$$x=-1-2h,$$

$$y=\frac{-2h^2d+3h+1}{b},$$

$$b=\frac{(-2h^2d+3h+1)^2}{4h^2d-4h-1} \quad (b\neq\infty).$$

Eliminating y from the equations of circle and conic gives a biquadratic whose roots are of course the abscissae of intersection points, one of which has been found above. The depressed equation should have a double root to give the desired tangency. The discriminant $\triangle$ of this cubic must vanish, thus giving an equation for the determination of h and thence of a and b in terms of e. For this cubic the coefficients are

$$a_0=d+1,\ 3a_1=h(-2d+2)+3d+7,$$

$$3a_2=\frac{h^3(16d^2)+h^2(-12d^2-56d)+h(8d+44)+3d+11}{N},$$

$$a_3=\frac{h^3(8d^2)+h^2(-4d^2-24d)+h(2d+18)+d+5}{N},$$

in which $N=1-4h^2d+4h$, also

$$\triangle\equiv4(a_0a_2-a_1^2)(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2=0.$$

This form of $\triangle$ leads to some saving of labor over that of the expanded form.

The following are given as aids in comparison of results:

$$a_0a_2-a_1^2=\frac{4}{9N}[h^4(4d^3-8d^2+4d)+h^3(-8d^2+36d-4)+h^2(2d^2+25d-29)$$
$$+h(d-23)-4]$$

$$-(a_1a_3-a_2^2)=\frac{4}{9N^2}[h^6(16d^4+48d^3)+h^5(-112d^3-192d^2)$$
$$+h^4(4d^3+236d^2+252d)+h^3(20d^2-188d-108)$$
$$+h^2(-2d^2-57d+49)+h(-d+35)+4]$$

$$a_0a_3-a_1a_2=\frac{8}{9N}[h^4(4d^3-4d^2)+h^3(-16d^2+14d)+h^2(2d^2+31d-11)$$
$$+h(d-21)-4].$$

The final equation giving h, is

$$16h^8d^4(d-1)^2-16h^7d^3(4d^2-17d+5)-4h^6d^2(4d^3-31d^2+219d-37)$$

$$+4h^5d(10d^3-80d^2+337d-30)+h^4(4d^4-40d^3+682d^2-997d+36)$$

$$-h^3(4d^3-92d^2+679d-285)+h^2(2d^2-147d+241)-h(9d-71)+7=0.$$

When $d=0$, corresponding to the cardioid, only one of the four roots is available, two of the others being excluded since $b\neq 0$ and the fourth one because $b\neq\infty$, already included in the second special case.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

245. Proposed by S. I. JONES, A. B., Gunter Bible College, Gunter, Texas.

The shell of a hollow iron ball is 4 inches thick, and contains $\frac{1}{5}$ of the number of cubic inches in the whole ball. Find the diameter of the ball.

I. Solution by S. A. COREY, Hiteman, Iowa.

Let r be the radius of the ball; $(r-4)$ will then be the radius of the hollow sphere enclosed by the shell. As the volumes of spheres are proportional to the cubes of their radii, the conditions of the problem require that

$r^3-(r-4)^3=\frac{1}{5}r^3$, or $\frac{4}{5}r^3=(r-4)^3$, whence, $r=\dfrac{4}{1-\sqrt[3]{\frac{4}{5}}}=55.79$ inches, nearly.

II. Solution by M. R. BECK, Cleveland High School, Ohio.

Let r be the radius of the ball, then $\frac{4}{3}\pi r^3=\frac{4}{15}\pi r^3+\frac{4}{3}\pi(r-4)^3$..........(1).
From (1) we have $r^3-60r^2+240r-320=0$..........(2).
Substitute $r=x+\dfrac{320}{x}+20$ in (2), $x^3+\dfrac{32,768,000}{x^3}-11520=0$..........(3).
Solving the quadratic (3), $x=\sqrt[3]{(6400)}$ or $\sqrt[3]{(5120)}$.
Either root makes $r=55.8016$, and the diameter is 111.6032 inches.

Also solved by P. S. Berg, G. W. Greenwood, A. H. Holmes, L. E. Newcomb, D. B. Northrup, J. Scheffer, J. E. Sanders, and G. B. M. Zerr.

AVERAGE AND PROBABILITY.

169. Proposed by HENRY HEATON, Atlantic, Iowa.

*What is the average length of all straight lines that can be drawn within a given square in every possible direction and every possible length from every point of the square; if all the lines are equally distributed about the starting point and equally distributed as to length.

*The problem as restated above is somewhat different from the one solved in our columns last month. As the above conveys the original meaning of the Proposer it is published as a third solution.